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A New Theory of Elementary Matter
Part III: A Self-Consistent Field Theory of Electrodynamics and Correspondence with Quantum Mechanics

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Abstract

This paper exploits the axioms and general mathematical structure of a new theory of elementary matter, thus far developed in two earlier papers (Sachs, 1971b, c). It is shown here, in an explicit fashion, how the exact form of this theory approaches that of quantum mechanics of a 'many-particle' system that interacts electromagnetically. The form of the mathematical expression of quantum mechanics of a many-particle system is found to be a linear approximation for the nonlinear (deterministic) field theory of this author's approach. The latter approximation is valid only when the components of the (asserted) closed system are sufficiently weakly coupled so that it appears as a many-particle system. The physical equivalent of the Pauli exclusion principle is derived in this paper as an *exact* feature of the theory, which is, in fact, sensitive to its closed and nonlinear features. It is then shown how the Fermi-Dirac statistics in particle physics follows from the present nonlinear theory only in a linear approximation.

1. *Introduction*

The *mathematical structure* of the most primitive representation of the matter field equations was demonstrated in Part II (Sachs, 1971c) in terms of a 2-component spinor formalism. This formalism, in turn, implied an equivalent representation in terms of a positive-definite mass field [Part II, equations (2.20), (2.26)] that couples time-reversed spinor variables.

The coupling term in these equations that yields their nonlinear structure was denoted by

$$\mathscr{I}_i(\eta^{(1)}, \eta^{(2)}, \ldots \eta^{(i-1)}, \eta^{(i+1)}, \ldots \eta^{(n)})\, \eta^{(i)}$$

for the ith coupled field $(\eta^{(i)}, \chi^{(i)})$. If, now, this coupling should become sufficiently weak so that we can *approximate* it by zero, then the coupled field equations approach the form of a set of separated equations:

$$\begin{aligned} &\sigma_\mu\, \partial_i{}^\mu \eta^{(i)}(x_i) + \lambda^{(i)} \chi^{(i)}(x_i) = 0 \\ &\tilde{\sigma}_\mu\, \partial_i{}^\mu \chi^{(i)}(x_i) + \lambda^{(i)} \eta^{(i)}(x_i) = 0 \\ &(\partial_i{}^\mu \equiv \partial/\partial x_i{}^\mu) \qquad (i = 1, 2, \ldots n) \end{aligned} \tag{1.1}$$

Here, we have expressed each of the field solutions $(\eta^{(i)}, \chi^{(i)})$ as a function of the four space-time parameters x_i (for each of the field solutions) since the

vanishing of the coupling term. $\mathscr{I}_i \eta^{(i)}$, allows one to consider each the resulting uncoupled equations, *one at a time.* It is then perhaps more convenient to express the uncoupled equations (1.1) in terms of the four-component spinor formalism—since it is the part of $\mathscr{I}_i$ which destroys reflection symmetry that makes the more general form (in terms of 2-component spinors) more useful. Thus, according to the latter structure [Part II, equation (2.6)], equation (1.1) takes the form:

$$(\gamma_\mu \partial_i{}^\mu + \lambda^{(i)})\psi(x_i) = 0 \qquad (1.2)$$

$$(i = 1, 2, \ldots n) \qquad (\partial_i{}^\mu = \partial/\partial x_i{}^\mu)$$

where the Dirac matrices γ_μ were defined in the previous paper.

Equations (1.1) and (1.2) are linear differential equations. That is to say, the solutions of these equations are basis functions of operators that are independent of the solutions themselves. Nevertheless, it is important in this theory that since the coupling term $\mathscr{I}_i$ is never really independent of the field solution, even though it can be arbitrarily close to this situation, equations (1.1) and (1.2) cannot be considered as more than a *linear approximation* for nonlinear field equations.

Since each of the n spin or equations (1.2) is a *linear* differential equation, the sum of all n of these equations can be taken, yielding the following equivalent single field equation:

$$\left[\sum_{j=1}^{n} \gamma_\mu \partial_j{}^\mu + \lambda^{(j)}\right] \Psi(\psi^{(1)}(x_1), \psi^{(2)}(x_2), \ldots \psi^{(n)}(x_n)) = 0 \qquad (1.3)$$

The solution of this equation, Ψ, is a limiting form of a functional which, generally, represents a connective relation between the members of the set of solutions $\{\psi^{(i)}(x_i)\}$ of the coupled nonlinear equations. In this linear limit, Ψ has the form

$$\Psi = \prod_k \psi^{(k)}(x_k) \qquad (1.4)$$

The form of the solution (1.4) of equation (1.3) is, of course, only one out of $n!$ physically equivalent solutions of this equation, since the solution is independent of the order of factors, and there are $n!$ permutations of this order.

Because the differential equations for each of the n equations (1.2) are functionally identical, the complete set of solutions of each of these equations must span the same function space. Further, the required square integrability of the solutions of these equations implies that they must have the eigenfunction structure of ordinary linear quantum mechanics. It should be noted that the imposition of square integrability, i.e.

$$\int \psi^{(k)\dagger} \psi^{(k)} \, d\mathbf{r} = \text{finite no.}$$

is *generally* required by this theory, independent of whether or not the linear approximation is being used. The reason is based on the idea of 'conservation

of interaction' as a conceptual ingredient of the theory (see Part I, Sachs, 1971b).

The eigenfunction structure of the equations (1.2) then leads to the prediction of discreteness in the values of the (limiting forms of) the physical properties of the material system described.

With the limiting form of these equations in terms of eigenfunction solutions, the indices (k), for each of the field solutions, then stand for the set of quantum numbers (m_k) that relate to the elements of the Hilbert space of solutions of each of these equations. If we now take account of the fact that the order of factors in the solution (1.4) does not affect the predicted properties of the system (when this asymptotic form of the *nonlinear* theory is accurate), we may take a linear combination of all permutations of such product functions, with the assumption that each of the terms in this combination has equal weighting in describing the total interaction. The general solution of equation (1.3) then takes the form:

$$\Psi = \frac{1}{\sqrt{n!}} \sum_{P=1}^{n!} \exp(i\alpha_P) \prod_k \psi^{(m_k)}(x_k) \tag{1.5}$$

where α_P is a phase that is associated with the Pth permutation in the order of these product functions. Its value will be determined later on from the exact form of the function which fully exploits the nonlinear structure of the equations.

2. *Coupling to an External Potential*

The *uncoupling* of the matter field equations [Part II, (2.26)] was based on the assumption that each link, $\mathscr{I}_k \psi^{(k)}$, can be neglected in each of these equations, as compared with the other terms. This is equivalent to the requirement that the energy-momentum transfer within the components of the closed system is much smaller than the 'intrinsic energy' associated with each of these components. The latter is in terms of the free energy-momentum, expressed in terms of the operator $\gamma_\mu \partial_k{}^\mu$, and the 'rest energy', represented with the term $\lambda^{(k)}$. But the limit $\mathscr{I}_k \psi^{(k)} \to 0$ is really not required to achieve this uncoupling. It is only necessary that the following limit should be approached

$$\mathscr{I}_k \psi^{(k)} \to f(x_k) \psi^{(k)}$$

where $f(x_k)$ is an integrable function of the coordinates, but expressible in a form that is not explicitly a function of the field solutions $(\psi^{(1)}, \psi^{(2)}, \ldots \psi^{(k-1)}, \psi^{(k+1)}, \ldots \psi^{(n)})$ that couple to $\psi^{(k)}$. As we have seen earlier, this limit is analogous to the 'independent particle model' of nuclear theory, where $f(x_k)$ plays the role of a background potential (averaged over all nucleons except the kth one) that acts on the kth nucleon.

It should be emphasized that even with the linear approximation that has been discussed, the manifestations of the nonlinearity of the theory still appear in the predictions for observables. For example, the *finite width* for

the measured values of all observables (rather than the zero width predicted by actual eigenvalue theory) is a manifestation of the nonlinear features of the equations that involve the coupling of matter. That the width of the measured 'peaked' values of the properties of matter in the microscopic domain is never really zero, in principle, is, of course, in agreement with the actual observations. Even in the microdomain, the set of measured values of any physical property of a realistic system will always have the feature, that no matter how closely any two values of some property can be determined, there is always, *in principle*, a potentially measurable set of values (of this property) in between them.

The quantum theory interprets this 'natural width' in terms of intrinsically probabilistic notions (as incorporated in the Heisenberg uncertainty principle). On the other hand, the present theory leads directly to the natural width from the (nonlinear) mathematical terms in the field equations that have to do with the coupling of any given matter field component to its environment. Where the two theories should differ most strikingly is in the high energy limit, since the quantum theory rests on the axiom of linearity—under all experimental conditions—while this theory implies a continuity in the measured values of the properties of matter as a feature of the nonlinear aspects of the basic equations.

3. *Electrodynamics in Special Relativity*

Let us now consider the specific coupling of matter fields that is associated with the motion of electrically charged matter. This is the subject of electrodynamics. We have seen in the preceding section that the most primitive form of the electromagnetic field equations in special relativity is in terms of a 2-component spinor representation of the underlying (Poincaré) group. [Part II, equation (1.5)]. These field equations, in a generally covariant form (Sachs, 1964a), along with the accompanying field equations in the matter variables and the metric field variables (in general relativity) correspond to an extremum of the general action functional

$$A = A_D + A_M + A_L + A_E$$

The portion of A that yields the spinor form [Part II, equation (1.5)] of the electromagnetic equations (in special relativity) is

$$A_M = \left[ig_M \int \sum_{\alpha=1}^{2} a_\alpha \varphi_\alpha{}^\dagger (\sigma_\mu \partial^\mu \varphi_\alpha - 2Y_\alpha) + \text{h.c.} \right] d^4x \qquad (3.1)$$

when it is varied with respect to the spinor field variables φ_α, $\varphi_\alpha{}^\dagger$ and their first derivatives in space and time. g_M is a real constant that necessarily appears in the Lagrangian formalism. It cancels in the resulting field equations which in the Lagrangian form is the spinor equation

$$\partial \mathscr{L} / \partial \varphi_\alpha{}^\dagger - \partial_\mu [\partial \mathscr{L} / \partial (\partial_\mu \varphi_\alpha{}^\dagger)] = 0$$

and its conjugate equation. The coefficients a_α are the relative phases of the spinor equations for $\alpha = 1$ and 2, as they appear in the Lagrangian density. This will be found in the analysis of electron-positron pairs [to be carried out in Part IV (Sachs, 1971d)] to have the form $(-1)^\alpha$.

Since the dimension of the integrand in equation (3.1) is energy density per length, the fundamental constant g_M must have the dimension of length. It is an additional universal constant that *necessarily* appears in this theory as a consequence of the factorization of Maxwell's equations into a spinor form. It will appear in an interaction that determines the matter field solutions. Its magnitude will be found later on (in Part IV).

The action functional A_D is that part of A that contributes to the structure of the matter equations [Part II, equation (2.26)]. In special relativity, it has the form

$$\left[i\hbar c \int \eta^{(j)\dagger}(x)(\sigma_\mu \partial^\mu + \lambda^{(j)})\eta^{(j)}(x) + \chi^{(j)\dagger}(x)(\tilde{\sigma}_\mu \partial^\mu + \lambda^{(j)})\chi^{(j)}(x) + \text{h.c.}\right] d^4x \tag{3.2}$$

for each of the matter constituents (j) of the closed system. In terms of the 4-component bispinor formulation, A_D then has the following form for the entire closed system:

$$A_D = \left[i\hbar c \sum_j \int \bar{\psi}^{(j)}(x)(\gamma_\mu \partial^\mu + \lambda^{(j)})\psi^{(j)}(x) + \text{h.c.}\right] d^4x \tag{3.3}$$

The number ($\hbar c$) is, again, a constant that necessarily appears in the Lagrangian formalism. Its magnitude is determined from the conserved properties of a microscopic system which is described in terms of the spinor field variables $\psi^{(j)}$ (or $(\eta^{(j)}, \chi^{(j)})$. It is at this point where Planck's constant is introduced into the theory. (Henceforth we shall use units with $\hbar = c = 1$.)

The action functional A_L in A is chosen, for empirical reasons, to give the observed Lorentz force between interacting charged particle currents, in electrodynamics. In special relativity theory, with the bispinor formulation, it has the following form:

$$A_L = \sum_{\substack{j,k \\ (j \neq k)}} e^{(j)} e^{(k)} \int \bar{\psi}^{(j)} \gamma_\mu \psi^{(j)}(x) \int \bar{\psi}^{(k)} \gamma_\mu \psi^{(k)}(x') S(x - x') d^4x' d^4x \tag{3.4}$$

where, in this contribution to A, $e^{(j)} e^{(k)} = \pm e^2$ is the fundamental constant. $S(x - x')$ is the Green's function for D'Alembert's equation, i.e.

$$\Box S(x - x') = 4\pi\delta(x^\mu - x^\mu) \tag{3.5}$$

The action function A_L is the same as the conventional one, (that yields the Lorentz force) except for the exclusion of the self-energy terms in the summation (i.e. the terms with $j = k$). These are excluded here as a requirement of the conceptual basis of the theory (as discussed in Part I).

It is well known that the solutions of equation (3.5) are not unique—there

are the Green's function solutions S_{adv} and S_{ret}, corresponding to the advanced and retarded potentials of electromagnetic theory; and any linear combination of these is also a valid solution. The conventional approach to electrodynamics takes only the retarded Green's function as applicable because of the further restriction of 'causality'. The latter restriction, however, is imposed because of the interpretation of the matter field solutions in terms of particles—one of the interacting particles (j) being the 'emitter' and the other (k) the 'absorber'. On the other hand, the elementary interaction approach considers 'emitter' and 'absorber' as components of a single closed system—that is *in principle* without separable, distinguishable parts. With this approach, there are not separate times. There is only one space-time for the description of the entire closed system. The causality argument of the conventional particle theory does not then apply here. Instead, this theory requires that there should be no difference in the description of the elementary interaction should the labels 'emitter' and 'absorber' be interchanged. Thus, if 'emitter-absorber' is equivalent to 'absorber-emitter', it follows that the Green's function in A_L must *necessarily* be symmetric in the retarded and advanced terms, i.e.

$$S(x-x') = \tfrac{1}{2}(S_{ret} + S_{adv})$$
$$= \frac{1}{2|\mathbf{r}-\mathbf{r}'|}\{\delta[(t-t') - |\mathbf{r}-\mathbf{r}'|] + \delta[(t-t') + |\mathbf{r}-\mathbf{r}'|]\} \quad (3.6)$$

The action functional A_E is that part of A that yields the field equations in the metric field $q^\alpha(x)$—leading to a form that incorporates Einstein's gravitational field. This part of the analysis is not pertinent to our present discussion of electrodynamics in special relativity, except that, as we have pointed out earlier, the Pauli matrices σ^α (and in the bispinor formulation, the Dirac matrices which are constructed from σ^α) as they appear in the matter field equations, are an asymptotic limit (for sufficiently small space-time separations) of the fields $q^\alpha(x)$. The latter are the solutions of a well-defined set of differential equations (Sachs, 1967b).

The variation of the total action functional, A, with respect to the matter field spinor variables, then yields the matter field equations, including the effect of the electrodynamic interaction. The matter field spinor variables appear in A_D, A_L, and A_M. In A_M they are implicit in the source field Y_α, [equation (3.1)]. The latter have the following form in special relativity, in terms of the bispinor variables:

$$Y_1^{(j)} = e^{(j)}\bar{\psi}^{(j)}\Gamma_1\psi^{(j)} = 4\pi i e^{(j)}\begin{pmatrix}\bar{\psi}^{(j)}(-\gamma_0 + i\gamma_3)\psi^{(j)} \\ \bar{\psi}^{(j)}(i\gamma_1 - \gamma_2)\psi^{(j)}\end{pmatrix} \equiv e^{(j)}\begin{pmatrix}\bar{\psi}^{(j)}\Gamma_1(1)\psi^{(j)} \\ \bar{\psi}^{(j)}\Gamma_1(2)\psi^{(j)}\end{pmatrix} \quad (3.7)$$

$$Y_2^{(j)} = e^{(j)}\bar{\psi}^{(j)}\Gamma_2\psi^{(j)} = 4\pi i e^{(j)}\begin{pmatrix}\bar{\psi}^{(j)}(-i\gamma_1 - \gamma_2)\psi^{(j)} \\ \bar{\psi}^{(j)}(\gamma_0 + i\gamma_3)\psi^{(j)}\end{pmatrix} \equiv e^{(j)}\begin{pmatrix}\bar{\psi}^{(j)}\Gamma_2(1)\psi^{(j)} \\ \bar{\psi}^{(j)}\Gamma_2(2)\psi^{(j)}\end{pmatrix}$$

In these equations, the factor $e^{(j)}$ is inserted only for convenience—noting that it is

$$e^{(j)}\, e^{(k)} = \pm e^2$$

that is the fundamental constant, rather than $e^{(j)}$.

It is important to recall at this point that the 2-component spinor variables Y_α transform in special relativity theory according to Part II, equation 1.6b and that these transformations are not form-invariant with respect to the vector variables j_μ (or $\bar{\psi}\gamma_\mu\psi$) of the standard representation of the theory. That this does not effect the predicted correspondence with conventional *observables* was discussed earlier in Part II.

If we now use the forms (3.1), (3.3) and (3.4) for the appropriate parts of the action functional and then vary the sum of these contributions with respect to the bispinor fields $\bar{\psi}^{(j)}$ (for the jth constituent matter component of the closed system) the following coupled field equations result:

$$(\gamma_\mu \partial^\mu - \mathscr{J}_j + \lambda^{(j)})\,\psi^{(j)}(x) = 0 \qquad (j = 1, 2, \ldots n) \tag{3.8}$$

where

$$\begin{aligned} \mathscr{J}_j = e^j \gamma_\mu \sum_{\substack{k=j \\ (k \neq j)}}^{n} \Big\{ & e^{(k)} \int \bar{\psi}^{(k)} \gamma_\mu \psi^{(k)}(x')\, S(x - x')\, d^4 x' \\ & + i g_M \sum_{\alpha=1}^{2} (-1)^\alpha \left[\varphi_\alpha^{(k)\dagger}(x) . \Gamma_\alpha - (\gamma_0 \Gamma_\alpha{}^\dagger \gamma_0) . \varphi_\alpha^k(x)\right] \Big\} \\ & \varphi_\alpha{}^\dagger . \Gamma_\alpha \equiv \varphi_\alpha{}^*(1)\, \Gamma_\alpha(1) + \varphi_\alpha{}^*(2)\, \Gamma_\alpha(2) \end{aligned} \tag{3.9}$$

It is an important feature of this theory that, *in its general form, all coupled fields $\psi^{(i)}$ are mapped in the same space-time x.* The matrices Γ_α are the combinations of Dirac matrices which are defined in equation (3.7). Recall that $\varphi_\alpha^{(k)}$ depends on the factor $e^{(k)}$ so that the products of terms in equation (3.9) are proportional to $e^{(j)} e^{(k)} = \pm e^2$.

The first term in equation (3.9) is the conventional interaction that is used in quantum electrodynamics—except for (a) the use of the Green's function (3.6) which is *symmetric* in the retarded and advanced terms (rather than only using the retarded term), and (b) the automatic rejection of self-energy terms in accordance with the axioms of this theory.

The second term, which depends on the universal length g_M, is a particular combination of the invariants $\varphi_\alpha{}^\dagger Y_\beta$ [Part II, equation (1.9)] that have no counterpart in the conventional Maxwell formulation of electromagnetism. Note that this interaction is not an *ad hoc* insertion; it is an extra contribution to the electromagnetic interaction resulting from a bona fide generalization of the theory which occurs when the vector representation of the theory is factorized into its irreducible spinor form. We will see later on that this term predicts the Lamb splitting in the fine structure of hydrogen, in agreement with the data. The magnitude of g_M will be determined from a comparison of the theoretical prediction with one of the Lamb splittings.

With this value, it will be found that the other Lamb splittings are in close agreement with the data. The justification for inserting $a_\alpha = (-1)^\alpha$ in equation (3.9) will come from the analysis of electron-positron pairs (in the following paper in this series).

It is interesting to note that the appearance of the generalized electromagnetic interaction (3.9) and its need to explain some physical phenomena, refutes the recently proclaimed 'principle of minimal electromagnetic coupling'.

4. *The Hartree-Fock Approximation*

Thus far, we have seen how the coupled nonlinear field equations of the elementary interaction field theory reduce to the structure of the quantum mechanical equations for a single particle system—either as a free particle or subjected to an external potential. It is now important to extend the analysis to the case of the 'many-body interaction' which could be applied, e.g. to the many-electron atom. At least from the empirical point of view, the spectra of such atoms are quite accurately described with the standard quantum mechanical formalism, with the Hartree and Hartree–Fock models, utilizing variational methods of calculation. It will be shown in this section that the nonlinear formalism of this theory reduces precisely to the Hartree formalism, when the nonrelativistic limit is taken. It will then be shown that the physical implications of the Pauli principle can be derived as a consequence of the *exact* formal structure of the field equations of this theory. The incorporation of this result with the Hartree approximation, in the non-relativistic limit, then leads naturally to the Hartree–Fock approximation for many-body systems that are in terms of spinor fields.

Taking the nonrelativistic limit of the uncoupled spinor field equations (1.2), the time part of each of the 4-vectors x_j then collapse into a common time coordinate, i.e.

$$v/c \to 0 \Rightarrow x_j \equiv (\mathbf{r}_j, t_j) \to (r_j, t)$$

and, at the same time, the bispinor solutions $\psi^{(j)}$, in the Dirac structure of the equations, approach the Schrödinger solutions ψ_S.

If one now wishes to improve the solutions of the uncoupled equations (1.2), when they are expressed nonrelativistically, the coupling functional may be re-introduced by inserting the *nonrelativistic* form of $\mathscr{I}_j$ into equation (3.8)—recalling that *in this limit only*, all field solutions may be described in their own spaces and a common time, i.e. $\psi^{(j)}(x) \to \psi(\mathbf{r}_j, t)$. The latter coupling term is then treated as a perturbation on the solutions $\psi(\mathbf{r}_j, t)$ of the uncoupled equations (1.2). In this way, the nonlinear, relativistically covariant formalism that we started with [equation (3.8)] reduces to the usual Hartree formalism. This result will now be demonstrated.

The nonrelativistic limit of the Dirac bispinor solutions are the Schrödinger solutions ψ_S. The added (generalized) part of the interaction

term (3.9) entails the mixing of the four components of $\psi^{(j)}$. (This will be shown in an explicit example in the analysis of the hydrogen spectrum in Part IV.) That is, the latter interaction term is expressed in terms of a nondiagonal operator. It then follows that in the Schrödinger limit, where three out of the four components of the Dirac solution may be neglected, the latter interaction term becomes ineffective.

Thus, the only part of the electrodynamic interaction that remains to perturb the (otherwise uncoupled) matter fields (in the Schrödinger limit) is the following nonrelativistic limit of the first part of $\mathscr{I}_j$:

$$\mathscr{I}_j(\mathbf{r}_j, t) = e^{(j)} \sum_{\substack{k \\ (k \neq j)}} e^{(k)} \int \psi_S^*(\mathbf{r}_k, t)\, \psi_S(\mathbf{r}_k, t)\, S(\mathbf{r}_j - \mathbf{r}_k)\, d\mathbf{r}_k \tag{4.1}$$

When all time coordinates are equated, the Green's function (3.6) takes the following form

$$S(\mathbf{r}_j - \mathbf{r}_k) = 1/4\pi|\mathbf{r}_j - \mathbf{r}_k| \tag{4.2}$$

With this substitution, the nonrelativistic form of equation (3.8) is expressed as follows:

$$\left\{ -\frac{1}{2m^{(j)}} \nabla_j^2 + e^{(j)} \sum_{\substack{k=1 \\ (k \neq j)}}^{n} e^{(k)} \int \frac{\psi_S^*(\mathbf{r}_k, t)\, \psi_S(\mathbf{r}_k, t)}{4\pi|\mathbf{r}_k - \mathbf{r}_j|} d\mathbf{r}_k \right\} \psi_S(\mathbf{r}_j, t) = i\partial^t \psi_S(\mathbf{r}_j, t) \tag{4.3}$$

$$(j = 1, 2, \ldots, n)$$

In the usual application of equation (4.3) to the properties of many-electron atoms, the solutions refer to 'stationary states'. In this case, the solutions factor into a spatial and temporal part:

$$\psi_S(\mathbf{r}_k, t) \to u_S(\mathbf{r}_k) \exp(-iE_k t)$$

With this substitution, equation (4.3) takes the form

$$\left\{ -\frac{1}{2m^{(j)}} \nabla_j^2 + e^{(j)} \sum_{\substack{k=1 \\ (k \neq j)}}^{n} e^{(k)} \int \frac{u_S^*(\mathbf{r}_k)\, u_S(\mathbf{r}_k)}{4\pi|\mathbf{r}_j - \mathbf{r}_k|} d\mathbf{r}_k \right\} u_S(\mathbf{r}_j) = E_j u_S(\mathbf{r}_j) \tag{4.3'}$$

$$(j = 1, 2, \ldots, n)$$

These are a set of n coupled differential equations for the n-particle system. Each set of solutions (there is an infinite number of such sets $(u(r_1), u(r_2), \ldots, u(r_n))_q$ for an n-body system, corresponding to the eigenvalues $(E_q, \ldots)$ for the various values of energy of the system. The formalism (4.3′)—a set of n coupled wave equations in the solutions $(u(r_1), u(r_2), \ldots)$—are the expression of the Hartree theory. It has been applied quite successfully to the prediction of properties of some many-body systems. In particular, the application of Hylleraas' variational method (Hylleraas, 1929) to determine the energy eigenvalues of the helium atom has yielded a close agreement with the observed properties of the spectrum of helium.

The physical system that is described by this Hamiltonian contains the Coulomb interaction operator that couples the nucleus to the jth electron field, with the (attractive) interaction constant $-Ze^2$, as well as all of the other (repulsive) Coulomb couplings between the remaining electrons of the atom and the jth electron.

Clearly, the eigenfunctions of the preceding operator will depend on all of the coordinate vectors $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{j-1}, \mathbf{r}_{j+1}, \ldots, \mathbf{r}_n)$, as well as the coordinates $\mathbf{r}_j$ of the jth electron. The energy eigenvalues of this operator must then be labelled, $E_{j;\,1,\,2,\,\ldots,\,j-1,\,j+1,\,\ldots,\,n}$.

The next step in determining the form of the many-particle eigenfunctions follows from the fact that each of the electron solutions (associated with each of the indices j) solves an eigenfunction equation that is functionally identical to each of the other equations. It then follows that the eigenfunctions of the operator which is the sum of all such operators, will satisfy the continuity equation:

$$\partial^t |u_S|^2 + \frac{\hbar}{2im} \nabla \cdot (u_S^* \nabla u_S - u_S \nabla u_S^*) = 0$$

We can then take the eigenfunctions of such a *sum of operators* (equation (4.3′)) to be the approximation for the weighting functions in the nonrelativistic description of the n-electron atom. In the structuring of these weighting functions, all of the electrons are treated on an equal footing. Only the binding to the nuclear Coulomb field is different. Assuming now that the nucleus is sufficiently inertial (relative to the electrons' dynamical properties) it can be considered as stationary:

$$|u_S(\mathbf{r}_N)|^2 \sim \delta(\mathbf{r}_N)$$

We may then treat $\mathbf{r}_N$ as a constant parameter to represent the distance from the origin of the coordinate system to the center of the binding nucleus.

5. *Another Approximation for the Many-Electron Atom*

In the previous example, as in the one to be discussed now, the atomic nucleus may be considered to be dynamically uncoupled from the electrons, because of the large nuclear mass, compared with the electron mass. This corresponds to neglecting the nuclear recoil that results from its absorption of energy-momentum from the bound electrons. However, since the electrons all have the same mass, one may not uncouple them from each other in the same way. It then follows that the weighting function which treats the electrons, one at a time, is perhaps not too good a representation of the true state of affairs (as far as the electron system is concerned)—even in the nonrelativistic form of the theory. For the many-electron atom, then, we wish to consider the better approximation

$$\{u_S(\mathbf{r}_1), u_S(\mathbf{r}_2), \ldots, u_S(\mathbf{r}_n)\} \to u_S(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n)$$

for the various states of the n-electron atom.

To arrive at the n-electron function on the right, we recognize that the

function $1/|\mathbf{r}_k - \mathbf{r}_j|$ in equation (4.3′) is a much more slowly varying function of $\mathbf{r}_k$ than is the (squared) amplitude $|u_S(r_k)|^2$. Thus we may use the following approximation without too much loss of accuracy:

$$\int \frac{|u_S(\mathbf{r}_k)|^2}{4\pi|\mathbf{r}_k - \mathbf{r}_j|} d\mathbf{r}_k \to \frac{1}{|\mathbf{r}_k - \mathbf{r}_j|} \int \frac{|u_S(\mathbf{r}_k)|^2}{4\pi} d\mathbf{r}_k = \frac{1}{|\mathbf{r}_k - \mathbf{r}_j|}$$

(assuming above that $\{u_S(r_k)\}$ are a normalized set of functions). The actual justification for this replacement, however (in the Hartree theory as well as the theory discussed below), must lie in the success of the predictions of the theory.

With the preceding replacement, then, equation (4.3′) takes the following form

$$\left\{-\frac{1}{2m^{(j)}}\nabla_j^2 - \frac{Ze^2}{|\mathbf{r}_N - \mathbf{r}_j|} + e^2 \sum_{\substack{k=1 \\ (k \neq j)}}^{n} \frac{1}{|\mathbf{r}_k - \mathbf{r}_j|}\right\} u(\mathbf{r}_j) = E_j u(\mathbf{r}_j)$$

One can go about trying to solve this equation, say, for helium, by starting with a set of trial solutions $u(\mathbf{r}_1)$, $u(\mathbf{r}_2)$, as in the Hartree theory, substituting them into the two coupled equations (4.3′) for this case, and then iterate toward a 'good solution'. An alternate way of attacking the helium problem, to yield solutions of the form $u(\mathbf{r}_1, \mathbf{r}_2)$, has been a method devised by Pekeris (1958, 1959). He starts with a trial function of three independent parameters $\psi(r_1, r_2, r_{12})$, solving the sum of two 2-particle operators, as described above. Thus,

$$\left\{-\frac{1}{2m}\nabla_1^2 - \frac{1}{2m}\nabla_2^2 + \left(\frac{1}{r_{12}} - \frac{Z}{r_1} - \frac{Z}{r_2} - E\right)\right\}\psi(r_1, r_2, r_{12}) = 0$$

where $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$. Making the change of variables

$$s = \epsilon(r_1 + r_{12} - r_2), v = \epsilon(r_2 + r_{12} - r_1), w = 2\epsilon(r_1 + r_2 - r_{12})$$

where $\epsilon = \sqrt{-E}$, and starting with the trial function (with the proper convergence properties):

$$\psi(s, v, w) = \exp\left[-\tfrac{1}{2}(s + v + w)\right] F(s, v, w)$$

the 2-body wave equation is then expressed in terms of these new variables. The method of solution consists in converting the problem of solving the two coupled differential equations, for the explicit solutions, into a problem of diagonalizing an (arbitrarily large) matrix—yielding the energy eigenvalues for the system. This is done by substituting the separated trial function

$$F(s, v, w) = \sum_{l,m,n} A(l, m, n) L_l(s) L_m(v) L_n(w)$$

(in ψ) where L_n is the Laguerre polynomial of order n, in the two-body wave equation. This, in turn, yields a set of recurssion relations between the coefficients $A(l, m, n)$, which are a set of linear algebraic equations. The vanishing of the determinant of these algebraic equations then yields the energy eigenvalues for the helium atom.

In this way, Pekeris showed that the energy eigenvalues of the low lying states of helium are very accurately predicted by a 2-body form of the wave equation (4.3′). It agrees with the variational calculation of Hylleraas—but it was carried out to more significant figures with the use of modern computing machine facilities. In any case, the close agreement between these calculations and the observed spectrum of helium further justifies the quantum mechanical expression of this system, along with the preceding approximations that were considered.

The fact that in the nonrelativistic limit, the elementary interaction field theory yields precisely *the same equation* as nonrelativistic quantum mechanics for the 'many-body' system, means that Pekeris' result is just as much a verification of this theory as it is of quantum mechanics. Nevertheless, when we wish to perfect these results by extension to the relativistic region, differences between the predictions of the elementary interaction field theory and quantum theory should occur. For in this case, the *nonlinear* features of the proposed theory, as well as the appearance of the additional (g_M-dependent) electrodynamic interaction, should predict deviations from what would be expected from the extension of the quantum theory to quantum field theory. The latter extension would be in terms of the Bethe–Salpeter theory (Bethe & Salpeter, 1957) for the 2-electron system, bound to a nucleus.

Nevertheless, the experimental data, to this time, reveals that the close agreement with Pekeris' (or Hylleraas') calculations (that includes the insertion of some 'relativistic terms') do not really require extension to the domain of quantum field theory in order to explain the low lying levels of helium. The study of the bound states of helium, then, is not a very good test that could distinguish between the elementary interaction field theory and quantum field theory. But this conclusion is not unexpected. It was implied earlier by the built-in feature of this theory that its formalism approaches that of ordinary quantum mechanics in the nonrelativistic limit.

Relativistic effects of bound systems, that do indeed follow from the present field theory, are the features of electron-positron pairs that relate to 'annihilation' and 'creation' processes (Part IV, Section 1) and the Lamb splitting in the fine structure of hydrogenic atoms (Part IV, Section 2). The latter consequences of the present formulation of electrodynamics will be seen to follow from the mathematical *derivations* of a finite and mathematically consistent formulation of a relativistic field formalism of the elementary interaction theory.

In addition to these relativistic effects, the possibility also exists that some of the features of nonrelativistic quantum mechanics that are normally *inserted* into this formalism when low energy physics is described, may be *derivable* from the exact form of one of the theories—elementary interaction field theory or quantum field theory—when they are properly expressed in the general form. Indeed, it will be shown below that one important feature of the nonrelativistic formalism—*the Pauli exclusion principle*—can be

derived from first principles, as a consequence of the exact nonlinear structure of the elementary interaction field theory.

6. *The Pauli Exclusion Principle* (Sachs, 1963)

In making the comparison between the proposed field theory of a closed system, and the standard quantum mechanical approach in terms of a many-body system, the following question naturally arises: How, within the framework of a pure field description of a closed system, does one interpret the Pauli exclusion principle, which appears to entail a correlation of the positions and momenta of different particles?

To answer the question, consider the physical implications of this principle. It asserts that two equivalent spin one-half particles cannot simultaneously be in the same state of motion—they cannot simultaneously be at the same location with the same constants of the motion. Thus the conventional form of the quantum theory places a restriction on the separate probability amplitudes $\psi_{(m_j)}(x_j)$ and $\psi^{(m_i)}(x_i)$ of the trajectories of particles (j) and (i). According to the elementary interaction field theory, on the other hand, there is only one space-time, x, and an interaction field amplitude $\Psi(x)$ that is mapped onto it. As we have indicated earlier, the interaction field amplitude is a connective relation between the component matter fields, $\{\psi^{(i)}(x)\}$—the solutions of the coupled nonlinear field equations that we start with.

One of the restraints on the interaction field amplitude $\Psi(x)$ is that it expresses the *law of conservation of interaction* (Part I) in terms of the continuity equation

$$\partial^{\mu}(\overline{\Psi}\gamma_{\mu}\Psi) = 0$$

The time-component $\overline{\Psi}\gamma_0\Psi(x) = \Psi^{\dagger}\Psi(x)$, whose integral over all space is then a constant in time, is a measure of the *weighting* of the mutual influence of all n components of the closed system, at the space-time point x. Within this interpretation it follows that if the physical situation should be approached that would correspond to an identical vanishing of the field amplitude Ψ at all space-time points, we would have to conclude that such a physical situation does not relate to any observable.

It will be shown below that if any two components of a physical system, identified with the indices (i) and (j), out of an n-component closed system, should have:

(1) a repulsive interaction, i.e. if $e^{(i)}e^{(j)} = +e^2$;
(2) the same inertial mass, i.e. $\lambda^{(i)} = \lambda^{(j)}$;
(3) the same state of motion, i.e.

$$\bar{\psi}^{(i)}\gamma_{\mu}\psi^{(i)}(x) = \bar{\psi}^{(j)}\gamma_{\mu}\psi^{(j)}(x) \qquad (6.1)$$

for all x;

then the contribution of the mutual coupling of (i) and (j) to the interaction weighting amplitude for the whole system must be identically zero.

The latter result is physically equivalent to the implications of the Pauli exclusion principle. It will then be shown that when this *exact* feature of the theory is incorporated with the nonrelativistic approximation for the many-particle interaction field amplitude, the latter function must vanish identically in the $4n$-dimensional space-time of the many-particle system. In particular, the latter asymptotic form of the interaction field will be shown to be the totally antisymmetrized Schrödinger wave function for the many-fermion system.

To proceed with this derivation, then, let Ψ_{ij} denote the interaction weighting amplitude for a system of coupled components, except for the contribution of the coupling $(i) \leftrightarrow (j)$. The remaining amplitude will be denoted by ψ_{ij}. To clarify this further, Ψ_{ij} can depend on the field amplitudes $\psi^{(i)}$ and $\psi^{(j)}$—but only with respect to their separate couplings to the other particle fields of the system. The amplitude Ψ_{ij} then omits the contribution of the *mutual coupling* of (i) and (j). If, then, we can find a field amplitude ψ_{ij} that vanishes under the three conditions mentioned above, then it would follow that $\Psi = \Psi_{ij}$ and the physical consequences of the Pauli principle would result.

Note that at this stage, we are only *assuming* that such a field ψ_{ij} exists. If no such 'two-body' field exists, then Ψ cannot be broken up into the two parts Ψ_{ij} and ψ_{ij}. Nevertheless, we will now see that such a field does indeed exist as a feature of the *exact form* of the coupled field equations.

To determine the form of ψ_{ij} for the special case under study, we first note that *if* $\bar{\psi}_{ij}\gamma_0\psi_{ij} = \psi_{ij}^\dagger\psi_{ij}$ represents an *additive contribution* to the weighting function for the closed system, then it must, by itself, satisfy an equation of continuity:

$$\partial^\mu(\bar{\psi}_{ij}\gamma_\mu\psi_{ij}) = 0 \tag{6.2}$$

We must then seek such a solution ψ_{ij}, that depends on the two matter-field solutions, $\psi^{(i)}$ and $\psi^{(j)}$ of the coupled set of nonlinear field equations (1.2) that satisfies a continuity equation (6.2) and vanishes identically under the special three conditions discussed above.

Multiplying the ith equation in (3.8) by the (conjugated) solution $\bar{\psi}^{(j)}$ of the jth equation, and multiplying the hermitian conjugate of the jth equation by $\gamma_0\psi^{(i)}$, subtracting these two equations and then repeating the operation with (i) and (j) interchanged, the sum of the two resulting equations gives:

$$\partial^\nu(\bar{\psi}^{(j)}\gamma_\nu\psi^{(i)} + \bar{\psi}^{(i)}\gamma_\nu\psi^{(j)}) =$$

$$(\bar{\psi}^{(i)}\gamma_\nu\psi^{(j)} - \bar{\psi}^{(j)}\gamma_\nu\psi^{(i)})R_\nu - ig_M\left\{\sum_{\alpha=1}^{2}\Phi_\alpha(\bar{\psi}^{(j)}\Gamma_\alpha\psi^{(i)} - \bar{\psi}^{(i)}\Gamma_\alpha\psi^{(j)}) - \text{h.c.}\right\}$$
$$+ (\lambda^{(j)} - \lambda^{(i)})(\bar{\psi}^{(j)}\psi^{(i)} - \bar{\psi}^{(i)}\psi^{(j)}) \tag{6.3}$$

where

$$R_\nu = e^{(i)}e^{(j)}\int(\bar{\psi}^{(j)}\gamma_\nu\psi^{(j)} - \bar{\psi}^{(i)}\gamma_\nu\psi^{(i)})S(x-x')\,d^4x'$$
$$+ (e^{(i)} - e^{(j)})\sum_{\substack{k \\ (k\neq i,j)}} e^{(k)}\left\{\int\bar{\psi}^{(k)}\gamma_\nu\psi^{(k)}S(x-x')\,d^4x'\right\} \tag{6.4}$$

and

$$\Phi_\alpha = (-1)^\alpha \{e^{(i)} \varphi_\alpha^{(j)\dagger} - e^{(j)} \varphi_\alpha^{(i)\dagger}\} + (e^{(i)} - e^{(j)}) \sum_{\substack{k \\ (k \neq i,j)}} \varphi_\alpha^{(k)\dagger} \tag{6.5}$$

The right-hand side of equation (6.3) contains three terms, each of them vanishing identically under some special conditions. First, if the ith and jth interacting components are each related to the same state of motion, i.e. under the special condition when equation (6.1) is true, then it also follows that

$$\bar{\psi}^{(i)} \Gamma_\alpha \psi^{(i)} = \bar{\psi}^{(j)} \Gamma_\alpha \psi^{(j)} \tag{6.6}$$

since Γ_α are linear combinations of the Dirac matrices. With equation (6.1) in equation (6.4), the first term in R_ν is automatically zero.

The spinor form

$$\sigma_\mu \partial^\mu \varphi_\alpha^{(i)} = e^{(i)} \bar{\psi}^{(i)} \Gamma_\alpha \psi^{(i)}$$

of the electromagnetic field equations implies that

$$\sigma_\mu \partial^\mu (e^{(j)} \varphi_\alpha^{(i)} - e^{(i)} \varphi_\alpha^{(j)}) = e^{(i)} e^{(j)} (\bar{\psi}^{(j)} \Gamma_\alpha \psi^{(j)} - \bar{\psi}^{(i)} \Gamma_\alpha \psi^{(i)}) \tag{6.7}$$

Since we can only accept the particular solutions of these equations (according to their interpretation within this theory (Parts I and II), the substitution of equation (6.6) into the right-hand side of equation (6.7) implies that the only solutions of this equation must correspond to the relation

$$e^{(j)} \varphi_\alpha^{(i)} - e^{(i)} \varphi_\alpha^{(j)} = 0 \tag{6.8}$$

Thus, with the ith and jth component fields relating to the same state of motion, the first part of Φ_α in equation (6.5) must vanish.

Second, if the inertial mass parameters of the ith and jth fields are the same, the last term on the right-hand side of equation (6.3) must also vanish.

Finally, the second term on the right-hand side of equation (6.4) and the second term on the right-hand side of equation (6.5) involve the coupling of the ith and jth field components, *separately*, to all of the other interacting components of the system. If the mutual coupling between the ith and jth fields is repulsive, i.e. if $e^{(i)} e^{(j)} = +e^2$, then it follows that the coupling of the ith and jth fields to the other fields would have the same sign, i.e. $e^{(i)} e^{(k)} = e^{(j)} e^{(k)}$. Thus, under these circumstances, the last terms on the right-hand sides of equations (6.4) and (6.5) also must vanish.

Summarizing, when any two, out of an n-component closed system, are (1) in the same state of motion, (2) have the same inertial mass, and (3) have a repulsive interaction, then R_ν, Φ_α and $(\lambda^{(i)} - \lambda^{(j)})$—and therefore the entire right-hand side of equation (6.3), *automatically vanishes*. The following equation then results:

$$\partial^\nu (\bar{\psi}^{(j)} \gamma_\nu \psi^{(i)} + \bar{\psi}^{(i)} \gamma_\nu \psi^{(j)}) = 0 \tag{6.9}$$

If we now combine this result with the continuity equation for each of the separate matter fields (which is, fundamentally, a consequence of gauge invariance of the first kind),

$$\partial^\mu(\bar{\psi}^{(i)}\gamma_\mu\psi^{(i)}) = \partial^\mu(\bar{\psi}^{(j)}\gamma_\mu\psi^{(j)}) = 0 \tag{6.10}$$

and the requirement of the elementary interaction theory that the description must, generally, be symmetric with respect to the interchange

$$\psi^{(i)}(x) \leftrightarrow \psi^{(j)}(x)$$

the following result is obtained:

$$\partial^\mu[\overline{(\psi^{(i)} \pm \psi^{(j)})}\gamma_\mu(\psi^{(i)} \pm \psi^{(j)})] = 0$$

Thus, the solution ψ_{ij} of equation (6.2) is

$$\psi_{ij}(\pm) = \psi^{(i)} \pm \psi^{(j)} \tag{6.11}$$

The only ambiguity that still remains in ψ_{ij} is the $+$ or $-$ case. If the interaction field is, generally, unique, then for the considered situation, only one of these signs can be valid, *under all conditions.* Let us now determine which is the valid sign.

Since both the ith and jth interacting fields correspond in this case to the same state of motion, and have the same separate interactions with the rest of the closed system (and also have identical boundary conditions imposed by the remainder of the closed system) it follows that in the special case where (6.11) is valid, $\psi^{(i)}$ and $\psi^{(j)}$, separately, solve identical field equations with the same boundary conditions. Thus,

$$\psi^{(i)}(x) = \psi^{(j)}(x)$$

Note that there are not different arbitrary constants multiplying each side of this equation because of the special conditions imposed on these solutions (in this particular case) and the fact that these solutions solve nonlinear differential equations.

Now if the plus sign in (6.11) is correct, then we must solve for the matter field equation $\psi_{ij}(+) = 2\psi^{(i)}$, to determine the interaction field amplitude. However, in this case, this is a solution of the field equation

$$\hat{O}(\psi^{(1)}, \psi^{(2)}, \ldots \psi^{(j)} = \psi^{(i)}, \ldots \psi^{(n)})\psi^{(i)} = 0 \tag{6.12}$$

The symbol $\hat{O}$ denotes an integral-differential operator, such as the one that appears in equation (3.8), except for the feature that equation (6.12) describes a particle to be interacting with itself (as well as with the other constituent elements of the closed system). The acceptance of the solution $\psi_{ij}(+)$ would then be logically incompatible with the initial premise of the elementary interaction theory since, as we have argued earlier (Parts I and II) such an approach prohibits the appearance of any self-interaction terms. To be logically consistent with the initial premise of the theory, we are then required to take *only* the solution

$$\psi_{ij}(-) = \psi^{(i)} - \psi^{(j)} = 0 \tag{6.13}$$

This is the result that was to be proven. It implies that under the special conditions on the ith and jth interacting components of the closed system that led to the result in equation (6.13), the interaction field amplitude $\Psi(x)$ for the closed system reduces to the interaction field amplitude, Ψ_{ij}, which excludes any contribution from the *mutual coupling* of the ith and jth components of the system. It is then concluded that if any two, (i, j), out of the totality of mutually interacting components of a closed system, should approach the set of circumstances that correspond to their being in the same state of motion, having the same inertial mass, and having a repulsive interaction, then there is no possible measurement that can relate to the mutual coupling of the ith and jth components of the system. This *derived* conclusion is equivalent to the statement of the *Pauli exclusion principle.*

7. *Fermi-Dirac Statistics from the Nonrelativistic Approximation of* Ψ

Let us now return to the nonrelativistic approximation (1.5) for the interaction field amplitude. In this limit the time coordinate becomes a common parameter for all field components and any two factors in the products, when they correspond to equivalent states of motion, contribute the product

$$\psi_{ij}^{(0)} = \psi^{(m_i)}(\mathbf{r}_i)\,\psi^{(m_j=m_i)}(\mathbf{r}_j = \mathbf{r}_i) \tag{7.1}$$

If i and j should represent interacting particles with equal masses and mutually repelling forces, then the *exact* result that was derived in the preceding paragraph implies that the product (7.1) is *only an approximation for the exact solution* $\psi_{ij}(-)$. Since the latter was found to be identically zero in all space-time, the function $\psi_{ij}^{(0)}$, in equation (7.1), must actually be a non-zero approximation for zero! Finally, since the product function (7.1) is a factor that multiplies the product of all other matter fields in the product (1.5) for the closed system (in this asymptotic limit), it follows that the actual vanishing of $\psi_{ij}^{(0)}$ causes the total interaction field amplitude to vanish under the same physical circumstances.

To incorporate this result into the asymptotic interaction field amplitude for a system of equivalent 'particles', we must choose the phase in equation (1.5) to be

$$\alpha_P = \pm\pi P \Rightarrow \exp(i\alpha_P) = (-1)^P$$

In this case, then, an equivalent way to express this approximation for the interaction field amplitude is:

$$\Psi = \frac{1}{\sqrt{n!}} \sum_P (-1)^P \prod_{i=1}^{n} \psi^{(m_i)}(\mathbf{r}_i) = \frac{1}{\sqrt{n!}} \begin{vmatrix} \psi^{m_1}(\mathbf{r}_1) \dots \psi^{(m_1)}(\mathbf{r}_n) \\ \vdots \qquad\qquad \vdots \\ \psi^{(m_n)}(\mathbf{r}_1) \dots \psi^{(m_n)}(\mathbf{r}_n) \end{vmatrix} \tag{7.2}$$

which is the usual Slater determinant for the many-body wave function. This is the totally antisymmetrized wave function for the many-particle system (of spin-one-half particles) that leads to the Fermi-Dirac statistics

of an ensemble of indistinguishable objects, implying, e.g., the ordering in the periodic table, the properties of metals, etc.

Even though the interaction field amplitude (7.2) is identical with the many-body wave function for a system of noninteracting, indistinguishable particles (fermions), it should be noted here that the interpretation of this theory is not in terms of particles (separate 'things'). The statistical aspect appears in this theory only because we are using a limiting formalism in which the coupled fields that describe interacting matter, *appear to be uncoupled in a first approximation.* The Pauli exclusion principle, on the other hand, deals with the features implied by a particular state of mutual interaction—independent of approximation.

In quantum field theory, the Pauli principle automatically implies Fermi–Dirac statistics—in an exact sense. This is because the latter is a particle theory from the outset. In the present interaction field theory, on the other hand, the individual particle aspects are only the manifestation of an asymptotic description of a *closed system*, corresponding to the (unattainable) limit of no interaction between the component matter fields of the system. Thus, Fermi–Dirac statistics appears within this theory only as an approximation to describe a closed system in terms of fields that are very weakly coupled. This approximation should be valid only when the quantities of momentum-energy transfer within the closed system is nonrelativistic. Combining this requirement on the interaction field amplitude with the previously derived 'Hartree approximation', we have the Hartree-Fock formalism.

To sum up, it has been demonstrated in this paper how the general mathematical structure of this theory, shown earlier to be necessarily and uniquely implied by its axioms, leads to a formalism that approaches that of quantum mechanics, in the limit of sufficiently small energy-momentum transfer within the closed system that is asserted. Thus, all of the predictions of nonrelativistic quantum mechanics are also predictions of this theory. Yet this theory is conceptually different from the quantum theory—in contrast, this theory is based on the continuous field concept to describe a single closed system (rather than many particles), it is deterministic and it is fundamentally nonlinear.

An exact derivation from the formal structure of this theory leads to a result that is physically equivalent to the Pauli exclusion principle. It is important that this result is sensitive to features of this theory that are absent in the conventional approach (quantum mechanics). It is shown that the Fermi–Dirac statistics of 'particles' follows here as a linear approximation to the exact nonlinear theory.

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